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BOUNDS FOR THE t-TAIL AREA

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UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

BOUNDS FOR THE t-TAIL AREA

Andrew P. Soms

Technical Summary Report # 1806
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ABSTRACT

The bounds of Birnbaum (1942), Gordon (1941), Sampford (1953), and Tate (1953) for the upper tail area of the normal distribution are, with some modifications, extended to the t-distribution. Comparisons between the bounds are made and some numerical examples are provided.

AMS(MOS) Subject Classification: 60E05

Key Words: bounds, t-distribution, t-tail area.

Work Unit # 4 - Probability, Statistics and Combinatorics

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SIGNIFICANCE AND EXPLANATION

In order to use statistical tests it is necessary to have numerical values for the frequency distributions involved. These are usually obtained from published tables. For computer work it is often preferable to use simple approximate algebraic formulae rather than storing tables or doing numerical integration. Such formulae are often convenient also when using a hand calculator. Approximate formulae are particularly useful when upper and lower bounds can be given, bracketing the exact value. These remarks apply particularly to the "tails" of frequency distributions.

Bounds are well known for the normal distribution. This paper develops analogous bounds for the t-distribution, which is used, for example, when comparing means derived from two different sets of experiments.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

Bounds for the t-Tail Area

Andrew P. Soms

1. Introduction

There are many approximations known to t-tail areas (Johnson and Kotz 1970, Chapter 27). Bounds, however, do not seem to have been considered, even though they sometimes are more desirable. They are useful for bounding the descriptive level of the t-test as well as theoretical considerations. For the normal case bounds have been given in Johnson and Kotz (1970, Chapter 33). The purpose of this paper is to extend, with some modifications, the results of four papers on the normal distribution (Birnbaum 1942, Gordon 1941, Sampford 1953, and Tate 1953) to the t.

A brief review of the results for the normal distribution will provide some motivation for the t. Let

$$\phi(t) = (2\pi)^{-1/2} e^{-t^2/2} \text{ and } 1 - \Phi(x) = \int_x^\infty \phi(t)dt,$$

where here and throughout it will be understood that $x > 0$. Then in the normal case Mills' ratio R_x is defined by $R_x = (1 - \Phi(x))/\phi(x)$, and R_x is used to obtain bounds because it has the well-known asymptotic expansion

$$R_x = \sum_{i=0}^{\infty} (-1)^i \, 1\cdot 3 \cdots (2i-1)/x^{2i+1}, \tag{1.1}$$

with the property that if the first n terms of (1.1) are summed, then the error term (R_x-sum of first n terms) has the same sign as the first neglected term and is less than it in absolute value. For the t, let

$$f_k(t) = c_k(1 + t^2/k)^{-(k+1)/2}, \quad c_k = \frac{\Gamma((k+1)/2)}{\Gamma(k/2)(\pi k)^{1/2}},$$

k an integer ≥ 1, and

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$$\bar{F}_k(x) = 1 - F_k(x) = \int_x^\infty f_k(t)dt.$$

Here the correct definition of Mills' ratio R_x is

$$R_x = (\bar{F}_k(x)/((1+x^2/k)f_k(x)) = \bar{F}_k(x)/g(x)$$

(for simplicity the subscript k on g is omitted), since it was shown by Soms (1976) that R_x has the asymptotic expansion

$$R_x = \sum_{i=0}^{\infty} (-1)^i \frac{1\cdot 3 \cdots (2i-1)k^i}{(k+2)(k+4)\ \dots\ (k+2i)x^{2i+1}}\ , \tag{1.2}$$

with the same error property as for the normal, i.e., if (1.2) is summed to n terms, then the error term has the same sign as the first neglected term and is less than it in absolute value. In particular,

$$\frac{1}{x}(1 - \frac{k}{(k+2)x^2}) < R_x < \frac{1}{x}\ . \tag{1.3}$$

It will be seen that in general these bounds can be improved.

2. Extension of Birnbaum's Bound

By integration by parts,

$$\int_x^\infty t^{2n+1} f_k(t)dt = c_k \int_x^\infty t^{2n} \frac{k}{k-1} \frac{d}{dt} - (1 + \frac{t^2}{k})^{-(k-1)/2} dt =$$

$$\frac{k}{k-1}\ x^{2n}\ (1 + \frac{x^2}{k})\ f_k(x) + 2n\ \frac{k}{k-1} \int_x^\infty t^{2n-1}\ (1 + \frac{t^2}{k})\ f_k(t)dt,\ n \le \frac{k-2}{2}\ , \tag{2.1}$$

and

$$\int_x^\infty t^{2n} f_k(t)dt = \int_x^\infty t^{2n-1} \frac{k}{k-1} \frac{d}{dt} - (1 + \frac{t^2}{k})^{-(k-1)/2} dt =$$

$$\frac{k}{k-1} x^{2n-1} (1 + \frac{x^2}{k}) f_k(x) + (2n-1) \frac{k}{k-1} \int_x^\infty t^{2n-2} (1 + \frac{t^2}{k}) f_k(t)dt, \; n \leq \frac{k-1}{2} .$$

(2.2)

Hence if $k \geq 3$, letting $g(x) = (1 + x^2/k)f_k(x)$, $n = 0$ can be substituted in (2.1) and $n = 1$ in (2.2) to give

$$\int_x^\infty tf_k(t)dt = kg(x)/(k-1)$$

and

$$\int_x^\infty t^2 f_k(t)dt = kxg(x)/(k-1) + k/(k-1)\int_x^\infty (1 + t^2/k)\, f_k(t)dt,$$

or

$$((k-2)/(k-1)) \int_x^\infty t^2 f_k(t)dt = (k/(k-1))(xg(x) + \int_x^\infty f_k(t)dt),$$

or

$$\int_x^\infty t^2 f_k(t)dt = (k/(k-2))(xg(x) + \int_x^\infty f_k(t)dt).$$

The Cauchy-Schwartz inequality

$$(\int_x^\infty tf_k(t)dt)^2 < \int_x^\infty t^2 f_k(t)dt \int_x^\infty f_k(t)dt$$

gives

$$(kg(x)/(k-1))^2 < (k/(k-2))(xg(x) + \int_x^\infty f_k(t)dt)(\int_x^\infty f_k(t)dt),$$

or

$$\frac{(k)(k-2)}{(k-1)^2} g^2(x) < (\int_x^\infty f_k(t)dt + \frac{xg(x)}{2})^2 - \frac{x^2 g^2(x)}{4} ,$$

or

$$\left(\frac{(k)(k-2)}{(k-1)^2} + \frac{x^2}{4}\right)^{1/2} - \frac{x}{2} < \int_x^\infty f_k(t)dt/g(x) = \bar{F}_k(x)/g(x) , \tag{2.3}$$

which is the desired result.

Call the left-hand side of (2.3) the BL-bound. The known bound (the AL-bound) is

$$\frac{1}{x}\left(1 - \frac{k}{(k+2)x^2}\right) . \tag{2.4}$$

It is clear that near the origin the BL-bound is better (bigger is better for lower bounds), while an examination of the limit of the ratio of the AL-bound to the BL-bound as $x \to \infty$ shows it to be bigger than one and hence the AL-bound is better for large x.

3. Extension of Sampford's Bounds

In the course of subsequent development, some results on the c_k will be needed and they are collected here for easy reference. From Johnson and Kotz (1969, p. 6),

$$\Gamma(\alpha+1) = (2\pi)^{1/2}\,\alpha^{\alpha+1/2}\,e^{-\alpha+\theta_\alpha/12\alpha},\ 0 < \theta_\alpha < 1. \tag{3.1}$$

Using (3.1), upper and lower bounds on c_k are now obtained. For the upper bound, for $k > 2$,

$$c_k = \frac{\Gamma(\frac{k-1}{2}+1)}{\Gamma(\frac{k-2}{2}+1)(\pi k)^{1/2}} < \frac{((k-1)/2)^{(k-1)/2+1/2}\,e^{-(k-1)/2+1/6(k-1)}}{((k-2)/2)^{(k-2)/2+1/2}\,e^{-(k-2)/2}\,(\pi k)^{1/2}}$$

$$= \frac{k-1}{(k(k-2))^{1/2}}\,\frac{\left(1+\frac{1}{k-2}\right)^{\frac{k-2}{2}}}{e^{1/2}}\,\frac{e^{1/6(k-1)}}{(2\pi)^{1/2}} < \frac{(k-1)e^{1/6(k-1)}}{(k(k-2))^{1/2}(2\pi)^{1/2}} . \tag{3.2}$$

For the lower bound, for $k > 2$,

$$c_k > \frac{((k-1)/2)^{(k-1)/2 + 1/2}\, e^{-(k-1)/2}}{((k-2)/2)^{(k-2)/2 + 1/2}\, e^{-(k-2)/2 + 1/6(k-2)}\, (\pi k)^{1/2}}$$

$$= \frac{(k-1)(1 + 1/(k-2))^{(k-2)/2}}{(k(k-2))^{1/2}\, (2\pi)^{1/2}\, e^{1/2 + 1/6(k-2)}} . \tag{3.3}$$

Using (3.2) and (3.3) the following lemma is now proved.

Lemma 3.1: With c_k as above,

$$4c_k^2 < 1 \text{ or } c_k < \frac{1}{2}, \ k \geq 1, \tag{3.4}$$

$$-2c_k^2 + (k+1)/2k > 0 \text{ or } c_k < ((k+1)/k)^{1/2}/2, \ k \geq 1, \tag{3.5}$$

and

$$8c_k^2 > (k-1)/k \text{ or } c_k > (1/2)^{3/2}((k-1)/k)^{1/2}, \ k \geq 2. \tag{3.6}$$

Proof of (3.4): It is verified directly that (3.4) holds for $k=1$, 2, and 3. For $k \geq 4$, note that both $(k-1)/(k(k-2))^{1/2}$ and $e^{1/6(k-1)}$ are bigger than one and decreasing functions of k and hence it suffices to observe that the last expression of (3.2) evaluated for $k=4$ is $< \frac{1}{2}$.

Proof of (3.5): Since $((k+1)/k)^{1/2} > 1$, (3.5) follows immediately from (3.4).

Proof of (3.6): It is verified directly that (3.6) is true for $2 \leq k \leq 5$. For $k \geq 6$, using (3.3), it suffices to show that

$$\frac{(k-1)(1+1/(k-2))^{(k-2)/2}}{(k(k-2))^{1/2}(2\pi)^{1/2}e^{1/2 + 1/6(k-2)}} > \frac{(k-1)^{1/2}}{2^{3/2}k^{1/2}} ,$$

or, since $((k-1)/(k-2))^{1/2} > 1$, that

$$(1+1/(k-2))^{(k-2)/2} > \pi^{1/2}e^{1/2 + 1/6(k-2)}/2,$$

which is true for $k=6$ and hence for $k \geq 6$ since the left-hand side is an increasing function of k and the right-hand side decreasing.

Let $\nu(x) = 1/R_x$. Then

$$\lambda(x) = \nu'(x) = \nu(x)(\nu(x) - \frac{k-1}{k}x)/(1 + x^2/k) \tag{3.7}$$

and the first result is that $0 < \lambda(x) < 1$ for $k \geq 1$ (here and throughout, "'" means the derivative). In the normal case, Birnbaum's inequality is equivalent to $\lambda(x) < 1$ - for the t this is not so - in fact, it will be shown that $\lambda(x) < 1$ gives a bound which is uniformly better than the BL-bound. Therefore, the method of proof (the Cauchy-Schwartz inequality) used by Sampford and Birnbaum for the normal case must be different for the t. It is instructive to give a different proof for the normal which generalizes to the t. For the normal case, $\nu'(x) = \lambda(x) = \nu(x)(\nu(x)-x)$ (recall that $\nu(x) = (2\pi)^{-1/2} e^{-x^2/2}/(1-\int_{-\infty}^{x}(2\pi)^{-1/2} \cdot e^{-t^2/2}dt)$). Suppose for some x, $\lambda(x) \geq 1$. Then there exists an x such that $\lambda(x) \geq 1$ and $\lambda'(x) = 0$ since $\lambda(0) = (2/(2\pi)^{1/2})^2 < 1$ and from (1.1), for arbitrary ε and x sufficiently large, $\lambda(x) \leq 1 + \varepsilon$. Now $\lambda'(x) = \lambda(x)(\nu(x)-x) + \nu(x)(\lambda(x)-1)$. But from (1.1) $\nu(x) > x$, and hence $\lambda'(x) > 0$, giving a contradiction. Returning to the t, we have

<u>Theorem 3.1</u>: Let $\lambda(x)$ be given by (3.7). *Then $0 < \lambda(x) < 1$ for $k \geq 1$.*

<u>Proof</u>: For convenience, $\lambda(x)$ and $\nu(x)$ will be denoted by λ and ν. From (1.3) $\nu > x$, and hence $\lambda > 0$. Suppose $\lambda \geq 1$ for some x. Then there exists an x such that $\lambda \geq 1$ and $\lambda' = 0$, since from (1.2) $\lim_{x\to\infty} \lambda = 1$ and from (3.4) $\lambda(0) = 4c_k^2 < 1$. Now

$$\lambda' = \frac{\nu}{(1+x^2/k)^2}(\nu - \frac{k+1}{k}x)(\nu - \frac{k-1}{k}x) + \frac{\nu}{1+x^2/k}\left[\frac{\nu}{1+x^2/k}(\nu - \frac{k-1}{k}x) - \frac{k-1}{k}\right]$$

$$= \frac{\nu}{1+x^2/k}\left[\frac{(\nu-(k-1)x/k)(2\nu-(k+1)x/k)}{1+x^2/k} - \frac{k-1}{k}\right]$$

$$= \frac{\nu}{1+x^2/k}\left[\lambda(2 - \frac{k+1}{k}\frac{x}{\nu}) - \frac{k-1}{k}\right] > \frac{\nu}{1+x^2/k}(\frac{k-1}{k})(\lambda-1) \geq 0,$$

and hence $\lambda' > 0$, a contradiction, giving the conclusion.

From Theorem 3.1,

$$\nu(\nu - (k-1)x/k) < 1 + x^2/k. \tag{3.8}$$

Completing the square in (3.8) gives

$$(\nu - (k-1)x/2k)^2 < 1 + ((k+1)x/2k)^2,$$

or since $\nu - (k-1)x/2k > 0$ from (1.2),

$$\nu < (k-1)x/2k + (1 + ((k+1)x/2k)^2)^{1/2},$$

or

$$[(k-1)x/2k + (1+((k+1)x/2k)^2)^{1/2}]^{-1} < \bar{F}_k(x)/g(x). \tag{3.9}$$

Call the left-hand side of (3.9) the SL-bound. Then the SL-bound is uniformly better than the BL-bound. It suffices, using (2.3), to show that, for $z = x/2$,

$$\frac{k-1}{k} z + (1 + (\frac{k+1}{k} z)^2)^{1/2} < (\frac{(k-1)^2}{(k)(k-2)} + \frac{(k-1)^4}{(k(k-2))^2} z^2)^{1/2} + \frac{(k-1)^2}{(k)(k-2)} z,$$

or

$$(1 + (1 + \frac{1}{k})^2 z^2)^{1/2} < (\frac{(k-1)^2}{(k)(k-2)} + \frac{(k-1)^4}{(k(k-2))^2} z^2)^{1/2} + (\frac{1}{k} + \frac{1}{k(k-2)})z.$$

Squaring both sides and noting that the cross-product term on the right is $> 2z^2/k$ gives the result. Consider now the SL-bound and the AL-bound. The SL-bound will be better if

$$\left[\frac{k-1}{2k} x + (1 + (\frac{(k+1)}{2k} x)^2)^{1/2}\right]^{-1} > \frac{1}{x} - \frac{k}{(k+2)x^3}. \tag{3.10}$$

Since $(1 + a^2)^{1/2} < a + 1/2a$, $a > 0$, (3.10) will hold if

$$(x + \frac{k}{(k+1)x})^{-1} > \frac{1}{x} - \frac{k}{(k+2)x^3},$$

which is true if $x < k^{1/2}$. Noting that $(1 + a^2)^{1/2} > a - 1/2a$ for large a, and repeating the above, shows that for large x the AL-bound is better.

To summarize, the SL-bound is better than the BL-bound for all positive x and for $x < \sqrt{k}$ better than the AL-bound, while for large x the AL-bound is better.

Recall that

$$\lambda' = \frac{\nu}{1+x^2/k}\left[\frac{(\nu-(k-1)x/k)(2\nu-(k+1)x/k)}{1+x^2/k} - \frac{k-1}{k}\right] =$$

$$= \frac{\nu}{1+x^2/k}\left(\phi - \frac{k-1}{k}\right),\ \phi = \frac{(\nu-(k-1)x/k)(2\nu-(k+1)x/k)}{1+x^2/k}. \qquad (3.11)$$

Then the second result which will give an upper bound is

<u>Theorem 3.2</u>: Let λ' and ϕ be as in (3.11). Then, for $k \geq 2$, $\lambda' > 0$ or, equivalently, $\phi > (k-1)/k$.

<u>Proof</u>: Suppose $\lambda' \leq 0$ for some x. Then there exists an x such that $\phi \leq (k-1)/k$ and $\phi' = 0$, since $\phi(0) = 8c_k^2 > (k-1)/k$ by (3.6) and $\lim_{x\to\infty} \phi(x) = (k-1)/k$, since $\lim_{x\to\infty} \nu/x = 1$ by (1.2). Now using (3.7),

$$\phi' = \frac{[(\lambda-(k-1)/k)(2\nu-(k+1)x/k) + (\nu-(k-1)x/k)(2\lambda-(k+1)/k)](1+x^2/k)-2(1+x^2/k)x\phi/k}{(1+x^2/k)^2}$$

$$= \frac{(\nu-2x/k)\phi-((k-1)/k)(2\nu-(k+1)x/k) + (\nu-(k-1)x/k)(2\lambda-(k+1)/k)}{(1+x^2/k)}. \qquad (3.12)$$

Suppose $\nu \leq 2x/k$. Then $x < \nu \leq 2x/k$, since $\nu > x$ by (1.3), and so $kx < 2x$, which is impossible since $k \geq 2$, and so $\nu > 2x/k$. Adding and subtracting $(\nu-2x/k)(k-1)/k$ to the numerator of (3.12), gives, after some simple algebra, $\phi' = [(\nu-2x/k)(\phi-(k-1)/k) + 2(\nu-(k-1)x/k)(\lambda-1)]/(1+x^2/k) < 0$, since $\phi \leq (k-1)/k$ by assumption and $\lambda < 1$ by Theorem 3.1, which is a contradiction and completes the proof.

From Theorem 3.2,

$$(\nu-(k-1)x/k)(2\nu-(k+1)x/k) > (k-1)(1+x^2/k)/k, \tag{3.13}$$

and completing the square in (3.13) gives

$$(\nu-(3k-1)x/4k)^2 > (8(k-1)/k + ((k+1)x/k)^2)/16, \tag{3.14}$$

and since $\nu > (3k-1)x/4k$, (3.14) gives, after some algebra,

$$4/\left[(3k-1)x/k + (8(k-1)/k + ((k+1)x/k)^2)^{1/2}\right] > \bar{F}_k(x)/g(x). \tag{3.15}$$

Call the left-hand side of (3.15) the SU-bound and $1/x$, the bound obtained from (1.3), the AU-bound. Then inspection shows that the SU-bound is always better (smaller) than the AU-bound.

4. Analogues of Tate's Bounds

The upper and lower bounds derived here are suitable, unlike the others in this paper, for x close to 0 - in particular they will be seen to have the right limiting behavior as $x \to 0$. For simplicity the subscripts of f_k and F_k will be omitted in this section. Then the first result is

Theorem 4.1: Let $h = F(1-F)-f^2/4c_k^2$, $k \geq 1$. Then $h > 0$.

Proof: Let $c = 1/4c_k^2$. Now $h' = f(1 - 2F) + 2c(k+1)xf^2/(k(1 + x^2/k))$, since $f' = -(k+1)xf/(k(1 + x^2/k))$, and hence $h'(0) = 0$. Also

$$h'' = -2f^2 + \frac{k+1}{k(1+x^2/k)}\left[-fx(1-2F) + \frac{2c}{1+x^2/k}\left(2f^2x^2\left(-\frac{k+1}{k}\right) + f^2\left(1 - \frac{x^2}{k}\right)\right)\right]$$

$$= -2f^2 + \frac{k+1}{k(1+x^2/k)}\left[-xh' + \frac{2cf^2}{1+x^2/k}\left(\left(-1 - \frac{2}{k}\right)x^2 + 1\right)\right], \tag{4.1}$$

and so $h''(0) = -2c_k^2 + 2cc_k^2(k+1)/k = -2c_k^2 + (k+1)/2k > 0$ by (3.5). Therefore $h' > 0$ near the origin and since $h(0) = 0$ and $\lim_{x\to\infty} h(x) = 0$, $h(x)$ must have a (possibly relative) maximum ahead of any (possibly relative) minima.

Recall that for a maximum, $h'' \le 0$, and for a minimum, $h'' \ge 0$. Suppose that x is an extremum of h. Then $h'(x) = 0$ and hence using (4.1),

$$h'' = \frac{2f^2}{(1+x^2/k)^2}\left[-(1+x^2/k)^2 - c\,\frac{k+1}{k}\,(1+\frac{2}{k})x^2 + c\,\frac{k+1}{k}\right]$$

$$= \frac{2f^2}{(1+x^2/k)^2}\left[-1 + c\,\frac{k+1}{k} - (\frac{2}{k} + c\,\frac{k+1}{k}\,(1+\frac{2}{k}))x^2 - \frac{x^4}{k^2}\right]. \tag{4.2}$$

If x is a minimum, then

$$-1 + c(k+1)/k \ge (2/k + c(k+1)(1+2/k)/k)x^2 + x^4/k^2, \tag{4.3}$$

while if x is a maximum, the inequality (4.3) is reversed. Therefore if h has any minima, the minima must precede any maxima, which is a contradiction, hence h has no minimum, and this proves the result.

So

$$F(1-F) > cf^2,$$

or

$$(1-(-(1+x^2/k)^{-(k+1)} + 1)^{1/2}/2g(x) < \bar{F}_k(x)/g(x). \tag{4.4}$$

Call *the* left-hand side of (4.4) the TL-bound. For x close to the origin the TL-bound is the best since it gives the correct limiting value of 1/2. For large x all the other lower bounds given are better than the TL-bound. This is shown by considering the limiting value of the ratio of the bounds as $x \to \infty$.

The next theorem will be used to obtain an upper bound on $\bar{F}_k(x)/g(x)$.

Theorem 4.2: Let $g = (2F-1)(1+x^2/k)f - (k-1)F(1-F)x/k$, $k \ge 2$. Then $g > 0$.

Proof: $g' = 2f^2(1+x^2/k) - (k-1)F(1-F)/k \tag{4.5}$

and

$$g'' = -4(k+1)f^2x/k + 4xf^2/k + (k-1)(2F-1)f/k$$

$$= -4f^2x + (k-1)(2F-1)f/k. \qquad (4.6)$$

Note that $g(0) = 0$ and $\lim_{x\to\infty} g(x) = 0$. Hence if there is an x such that $g(x) \leq 0$, then there exists an x such that g(x) has a minimum at x for which $g(x) \leq 0$, $g'(x) = 0$, and $g''(x) \geq 0$. Now at x, using the definition of g, (4.5), and (4.6), g'' may be written as

$$g'' = -4f^2x + \frac{k-1}{k}\,\frac{(g+(k-1)F(1-F)x/k)}{(1+x^2/k)}$$

$$\leqq -4f^2x + \left(\frac{k-1}{k}\right)^2 \frac{F(1-F)x}{(1+x^2/k)}$$

$$= -4f^2x + \frac{k-1}{k}\,\frac{2xf^2(1+x^2/k)}{(1+x^2/k)} < 0,$$

which is a contradiction. Hence $g > 0$.

Completing the square in $g > 0$ gives

$$(F + k(1+x^2/k)f/(x(k-1))-1/2)^2 > [k(1+x^2/k)f/(x(k-1))]^2 + \frac{1}{4},$$

or, after taking square roots and some algebra,

$$(1/2 + k(1+x^2/k)f/(x(k-1)) - (1/4 + [k(1+x^2/k)f/(x(k-1))]^2)^{1/2})/g(x) > \bar{F}(x)/g(x). \qquad (4.9)$$

Call the left-hand side of (4.9) the TU-bound. As for the TL-bound, near the origin the TU-bound is best since it gives the correct limiting value and for large x all the other upper bounds given here are better.

5. Analogues of Gordon's Bounds

Here R_x and its derivatives are considered. Let $k \geq 2$, and recall that, dropping the subscript x,

$$R = \bar{F}_k(x)/g(x) = \bar{F}_k(x)/((1+x^2/k)f_k(x)).$$

Then

$$R' = ((k-1)xR/k-1)/(1+x^2/k), \qquad (5.1)$$

and the first conclusion is that $R' < 0$, or $(k-1)xR/k < 1$, or $\bar{F}/g < (k/(k-1))(1/x)$, and this is true since the AU-bound is $1/x$ and therefore uniformly better. This bound has been derived differently by Pinkham and Wilk (1963). It is instructive to give a direct proof since this leads naturally to the second result. Note that $(k-1)xR/k < 1$ at $x=0$ and $\lim_{x\to\infty} (k-1)xR/k = (k-1)/k$. Hence if $(k-1)xR/k \geq 1$, then there exists an x for which $(k-1)xR/k \geq 1$ and $\frac{d}{dx}(k-1)xR/k =$ $(k-1)R/k + (k-1)((k-1)xR/k-1)/(k(1+x^2/k)) = 0$, but this is a contradiction since $(k-1)xR/k-1 \geq 0$ and $(k-1)R/k > 0$, giving the conclusion. Call $(k/(k-1))(1/x)$ the GU-bound for future reference. The next result will give a lower bound on $\bar{F}_k(x)/g(x)$.

<u>Theorem 5.1</u>: $R'' > 0$ for $k \geq 3$.

<u>Proof</u>: Using (5.1), $R'' = \dfrac{((k-1)/k)((k-2)x^2/k+1)R - (k-3)x/k}{(1+x^2/k)^2}$, and hence $R'' > 0$ is equivalent to $h(x) = R - \dfrac{(k-3)x/(k-1)}{1+(k-2)x^2/k} > 0$. Note that $h(0) > 0$ and $\lim_{x\to\infty} h(x) = 2/((k-1)(k-2)) > 0$. So if $h(x) \leq 0$ for some x then there exists an x such that $h(x) \leq 0$ and $h'(x) = 0$. Differentiating,

$$h'(x) = \frac{(k-1)xR/k-1}{1+x^2/k} - \frac{k-3}{k-1}\,\frac{(1-(k-2)x^2/k)}{(1+(k-2)x^2/k)^2}\,, \tag{5.2}$$

and adding and subtracting $\dfrac{(k-1)x}{(1+x^2/k)k}\left(\dfrac{(k-3)x/(k-1)}{1+(k-2)x^2/k}\right)$ and some algebra gives

$$h'(x) = \frac{(k-1)x}{(1+x^2/k)k}\left(R - \frac{(k-3)x/(k-1)}{1+(k-2)x^2/k}\right) + ((1+x^2/k)(1+(k-2)x^2/k)^2)^{-1}$$

$$\cdot\left\{\left(-\left(\frac{k-2}{k}\right)^2 + \left(\frac{k-2}{k}\right)\left(\frac{k-3}{k-1}\right)\right)x^4 + \left(-2\,\frac{k-2}{k} + \frac{k-3}{k} + \frac{(k-2)(k-3)}{k(k-1)} - \frac{k-3}{k(k-1)}\right)x^2\right.$$

$$\left. + \left(-1 - \frac{k-3}{k-1}\right)\right\} < 0, \tag{5.3}$$

since the first term on the right-hand side of (5.3) is ≤ 0 by assumption and the second negative since inspection shows every coefficient to be negative. This is a contradiction and concludes the proof.

$R'' > 0$ gives

$$\frac{(k-3)x/(k-1)}{1+(k-2)x^2/k} < \frac{\bar{F}_k(x)}{g(x)} . \tag{5.4}$$

Call the left-hand side of (5.4) the GL-bound and assume that $k \geq 4$, since for $k = 3$ (5.4) is trivially true. Comparison to the AL-bound shows that up to an x the GL-bound is better and then the AL-bound is better. Consider now the comparison to the SL-bound. The SL-bound is better for all x for which

$$(1 + \frac{k-2}{k}x^2) > \frac{k-3}{k-1}x(\frac{k-1}{2k}x + (1 + (\frac{(k+1)x}{2k})^2)^{1/2}). \tag{5.5}$$

After some algebra, (5.5) is equivalent to

$$(1 + (k-1)x^2/2k)^2 > ((k-3)x/(k-1))^2(1 + ((k+1)x/2k)^2) \tag{5.6}$$

and comparing the coefficients of x^2 and x^4 on both sides, (5.6) is seen to be true for all positive x. Hence the SL-bound is better than the GL-bound.

6. Numerical Comparisons

In Table 1, the bounds obtained in this paper are given for some selected k and x. A "-" indicates that the bound is negative.

1. Lower and Upper Bounds for $\bar{F}_k(x)/g(x)$

Bounds	Degrees of freedom								
	6			10			20		
	x								
	.5	1.5	3.0	.5	1.5	3.0	.5	1.5	3.0
BL	.761	.484	.292	.775	.495	.299	.779	.499	.302
(AL,AU)	(——,2.000)	(.444,.667)	(.306,.333)	(——,2.000)	(.420,.667)	(.302,.333)	(—— ,2.000)	(.397,.667)	(.300,.333)
(GL,GU)	(.257,2.400)	(.360,.800)	(.257,.400)	(.324,2.222)	(.417,.741)	(.285,.370)	(.365,2.105)	(.444,.702)	(.295,.351)
(SL,SU)	(.800, .984)	(.512,.543)	(.306,.311)	(.792, .955)	(.507,.533)	(.305,.309)	(.787, .934)	(.504,.526)	(.304,.307)
(TL,TU)	(.725,1.044)	(.160,.692)	(.0106,.394)	(.736,1.012)	(.177,.656)	(.00991,.367)	(.744, .989)	(.191,.632)	(.00885,.349)

It can be seen that for the cases considered the SL-bound is the best among the lower bounds and the SU-bound the best among the upper. It is thus of interest what kind of approximations can be obtained to $\bar{F}_k(x)$ using the S-bounds multiplied by g(x) as the approximation. This is done in Table 2 for selected degrees of freedom and known t percentile (Cramer, 1946, p. 560). Here L = (SL-bound)·g(x) and U = (SU-bound)·g(x).

2. S-Approximations to $\bar{F}_k(x)$

	Degrees of Freedom								
$\bar{F}_k(x)$	6			20			120		
	x	L	U	x	L	U	x	L	U
.1	1.440	.0956	.102	1.325	.0959	.101	1.289	.0959	.101
.05	1.943	.0487	.0505	1.725	.0486	.0503	1.658	.0487	.0502
.025	2.447	.0246	.0252	2.086	.0245	.0251	1.980	.0245	.0251
.010	3.143	.00989	.0100	2.528	.00988	.0100	2.358	.00988	.0100
.005	3.707	.00496	.00502	2.845	.00496	.00501	2.617	.00496	.00501
.0005	5.959	.000498	.000500	3.850	.000497	.000500	3.373	.000498	.000501

From Table 2 it appears that the approximation using the SU-bound gives good results if $\bar{F}_k(x)$ is less than or equal to .1.

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The bounds of Birnbaum (1942), Gordon (1941), Sampford (1953), and Tate (1953) for the upper tail area of the normal distribution are, with some modifications, extended to the t-distribution. Comparisons between the bounds are made and some numerical examples are provided.

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